

Degeneration of tame automorphisms of a polynomial ring

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Abstract

Recently, Edo-Poloni constructed a family of tame automorphisms of a polynomial ring in three variables which degenerates to a wild one. In this note, we generalize the example by a different method.

1 Introduction

For each commutative ring R , let $R[\mathbf{x}] = R[x_1, \dots, x_n]$ be the polynomial ring in n variables over R , and $\mathrm{GA}_n(R)$ the automorphism group of the R -algebra $R[\mathbf{x}]$. We express $\phi \in \mathrm{GA}_n(R)$ as $(\phi(x_1), \dots, \phi(x_n)) \in R[\mathbf{x}]^n$. We say that $\phi \in \mathrm{GA}_n(R)$ is *affine* if the total degrees of $\phi(x_1), \dots, \phi(x_n)$ are equal to one, *triangular* if $\phi(x_i)$ belongs to $R^*x_i + R[x_1, \dots, x_{i-1}]$ for each i , and *elementary* if there exist $1 \leq l \leq n$ and $q \in R[\{x_i \mid i \neq l\}]$ such that $\phi(x_l) = x_l + q$ and $\phi(x_i) = x_i$ for $i \neq l$. Let $\mathrm{TA}_n(R)$ be the subgroup of $\mathrm{GA}_n(R)$ generated by affine automorphisms and triangular automorphisms. We say that $\phi \in \mathrm{GA}_n(R)$ is *tame* if ϕ belongs to $\mathrm{TA}_n(R)$, and *wild* otherwise. If K is a field, then $\mathrm{TA}_2(K)$ is equal to $\mathrm{GA}_2(K)$ by Jung [4] and van der Kulk [5]. In 2004, Shestakov-Umirbaev [8] showed that the same does not hold when $\mathrm{char} K = 0$ and $n = 3$. Today, a number of elements of $\mathrm{GA}_3(K)$ are known to be wild thanks to the Shestakov-Umirbaev theory and its modification [6].

Recently, Edo-Poloni [1, §4] constructed $\phi \in \mathrm{GA}_3(\mathbf{C}[t])$ with the following properties, where t is a variable, and $(x_1, x_2, x_3) = (z, y, x)$ in their notation:

(A) ϕ factors as $\tau^{-1} \circ \epsilon \circ \tau$ in $\mathrm{GA}_3(\mathbf{C}[t, t^{-1}])$, where $\tau \in \mathrm{GA}_3(\mathbf{C}[t, t^{-1}])$ is triangular and $\epsilon \in \mathrm{GA}_3(\mathbf{C}[t])$ is elementary.

(B) ϕ_0 is the wild automorphism $\exp x_1^{2l}(x_1x_3 + x_2^{l+1})\delta_0$, where $l \geq 1$ and δ_0 is the $\mathbf{C}[x_1]$ -derivation of $\mathbf{C}[\mathbf{x}]$ defined by $\delta_0(x_2) = x_1$ and $\delta_0(x_3) = -(l+1)x_2^l$.

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Here, for each $\alpha \in \mathbf{C}$, let ϕ_α be the automorphism of $\mathbf{C}[\mathbf{x}] = (\mathbf{C}[t]/(t - \alpha)) \otimes_{\mathbf{C}[t]} \mathbf{C}[t][\mathbf{x}]$ induced by ϕ . From (A), it follows that ϕ_α is tame if $\alpha \neq 0$. Thus, a wild automorphism is obtained as a “limit” of tame automorphisms. Using this example, Edo-Poloni [1, Cor. 4.3] concluded that $\mathrm{TA}_3(\mathbf{C})$ is not closed in the ind-group $\mathrm{GA}_3(\mathbf{C})$. The purpose of this note is to generalize Edo-Poloni’s example by a different method. From our construction, we easily see why such a phenomenon occurs.

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2 Construction

Let R be any commutative \mathbf{Q} -algebra, and δ a *triangular* R -derivation of $R[\mathbf{x}]$, i.e., an R -derivation of $R[\mathbf{x}]$ such that $f_i := \delta(x_i)$ belongs to $R[x_1, \dots, x_{i-1}]$ for each i . Then, for each $h \in R[\mathbf{x}]$ with $\delta(h) = 0$, we can define $\phi := \exp h\delta \in \mathrm{GA}_n(R)$ by $(\exp h\delta)(q) = \sum_{l \geq 0} h^l \delta^l(q)/l!$ for $q \in R[\mathbf{x}]$. For each $\mathfrak{p} \in \mathrm{Spec} R$, let $\phi_{\mathfrak{p}}$ be the element of $\mathrm{GA}_n(\kappa(\mathfrak{p}))$ induced by ϕ , where $\kappa(\mathfrak{p})$ is the residue field of the localization $R_{\mathfrak{p}}$. We consider when $\phi_{\mathfrak{p}}$ is tame.

First, recall that, when $n = 3$, R is a field and $f_1 = 0$, we have $\phi \notin \mathrm{TA}_3(R)$ if and only if $f_2 \neq 0$, $h \notin R[x_1]$ and $\partial_{x_2}(f_3) \notin f_2 R[x_1, x_2]$ (cf. [7, Thm. 3.2.3 (and the preceding remark)]). Hence, we get the following theorem, where \bar{f} denotes the image of f in $\kappa(\mathfrak{p})[\mathbf{x}]$ for each $f \in R[\mathbf{x}]$.

Theorem 1. *Assume that $n = 3$ and $f_1 \in \mathfrak{p}$. Then, $\phi_{\mathfrak{p}}$ is wild if and only if $\bar{f}_2 \neq 0$, $\bar{h} \notin \kappa(\mathfrak{p})[x_1]$ and $\partial_{x_2}(\bar{f}_3) \notin \bar{f}_2 \kappa(\mathfrak{p})[x_1, x_2]$.*

To discuss the case where $f_1 \notin \mathfrak{p}$, assume that f_1 is not a nilpotent element of R , and let R' be the localization R_{f_1} . Since no confusion arises, we use the letter ϕ to denote the element of $\mathrm{GA}_n(R')$ induced by ϕ , and δ to denote the derivation of $R'[\mathbf{x}]$ induced by δ , whose kernel is denoted by $R'[\mathbf{x}]^\delta$. Then, the image of h in $R'[\mathbf{x}]$ belongs to $R'[\mathbf{x}]^\delta$. In this situation, there exists a triangular automorphism $\tau = (x_1, g_2, \dots, g_n)$ of $R'[\mathbf{x}]$ such that $R'[\mathbf{x}]^\delta = R'[g_2, \dots, g_n]$. Actually, since $\delta(x_1/f_1) = 1$, we have $R'[\mathbf{x}] = R'[\mathbf{x}]^\delta[x_1/f_1]$, and

$$\sigma : R'[\mathbf{x}] \ni q \mapsto \sum_{l \geq 0} \frac{\delta^l(q)}{l!} (-x_1/f_1)^l \in R'[\mathbf{x}]$$

is a homomorphism of R' -algebras satisfying $\sigma(R'[\mathbf{x}]) = R'[\mathbf{x}]^\delta$ and $\sigma(x_1) = 0$ (cf. e.g. [2, 1.3.21 and 1.3.23]). Now, let $p \in R'[x_2, \dots, x_n]$ be such that

$\tau(p) = h$ in $R'[\mathbf{x}]$. Then, in $\mathrm{GA}_n(R')$, we have

$$\begin{aligned}\tau^{-1} \circ \phi \circ \tau &= \exp(\tau^{-1} \circ (h\delta) \circ \tau) = \exp(\tau^{-1} \circ (\tau(p)\delta) \circ \tau) \\ &= \exp(p\tau^{-1} \circ \delta \circ \tau) = \exp pf_1\partial_{x_1} = (x_1 + f_1p, x_2, \dots, x_n) =: \epsilon,\end{aligned}\quad (*)$$

and so $\phi = \tau \circ \epsilon \circ \tau^{-1}$. Therefore, the following theorem holds for any n .

Theorem 2. *If $f_1 \notin \mathfrak{p}$, then $\phi_{\mathfrak{p}}$ is tame.*

It is interesting to note that the extension $\tilde{\phi} \in \mathrm{GA}_{n+1}(R)$ of ϕ defined by $\tilde{\phi}(x_{n+1}) = x_{n+1}$ is tame by Smith [9]. In fact, let $\tilde{\delta}$ be the extension of δ to $R[x_1, \dots, x_{n+1}]$ defined by $\tilde{\delta}(x_{n+1}) = 0$, and γ the elementary automorphism $(x_1, \dots, x_n, x_{n+1} + h)$. Then, $\rho := \exp x_{n+1}\tilde{\delta}$ is tame and $\tilde{\phi} = \gamma^{-1} \circ \rho^{-1} \circ \gamma \circ \rho$.

Finally, we construct $\phi \in \mathrm{GA}_3(\mathbf{C}[t])$ satisfying (A) and (B) using our method. Let $R = \mathbf{C}[t]$, $f_1 = t$, $f_2 = x_1$ and $f_3 = -(l+1)x_2^l$, where $n = 3$. Then, we have $R' = \mathbf{C}[t, t^{-1}]$. Observe that δ kills

$$g_2 := x_2 - x_1^2/(2t) \quad \text{and} \quad g_3 := x_3 + \sum_{i=0}^l c_i t^{-(i+1)} x_1^{2i+1} x_2^{l-i},$$

and that $\tau := (x_1, g_2, g_3) \in \mathrm{GA}_3(\mathbf{C}[t, t^{-1}])$ is triangular, where $c_0 = l+1$ and $c_1, \dots, c_l \in \mathbf{Q}$ are defined by $c_i(2i+1) = -c_{i-1}(l-i+1)$ by induction. Set

$$p = (c_l t^l / 2) ((2x_2)^{2l+1} + t(x_3/c_l)^2) \quad \text{and} \quad \epsilon = (x_1 + tp, x_2, x_3).$$

Note that $h := \tau(p)$ is killed by δ . In the following, we check $h \in \mathbf{C}[t][\mathbf{x}]$ and $h|_{t=0} = x_1^{2l}(x_1 x_3 + x_2^{l+1})$. Here, for each $q \in \mathbf{C}[t][\mathbf{x}]$, we denote by $q|_{t=0}$ the element of $\mathbf{C}[\mathbf{x}]$ obtained from q by the substitution $t \mapsto 0$. Then, it follows that $\phi = \exp h\delta$ satisfies (B). Since $\phi = \tau \circ \epsilon \circ \tau^{-1}$ by (*), (A) is also satisfied.

We may write $h = t^{l+1}x_3^2/(2c_l) + x_1^{2l+1}x_3 + q$, where $q \in x_2\mathbf{C}[t, t^{-1}][x_1, x_2] + tx_3\mathbf{C}[t][x_1, x_2]$. Since the monomial $x_1^{2l+1}x_3$ appears in h , the minimal integer r for which $t^r h$ belongs to $\mathbf{C}[t][\mathbf{x}]$ is nonnegative. If r is positive, then $h' := (t^r h)|_{t=0} = (t^r q)|_{t=0}$ belongs to $x_2\mathbf{C}[x_1, x_2] \setminus \{0\}$. Let δ_0 be the derivation of $\mathbf{C}[\mathbf{x}]$ as in (B). Then, we have $\delta_0(h') = x_1\partial_{x_2}(h') \neq 0$, while $\delta_0(h') = \delta(t^r h)|_{t=0} = 0$ since $\delta(t^r h) = 0$. This is a contradiction. Thus, we get $r = 0$, and so $h' = h|_{t=0} = x_1^{2l+1}x_3 + q|_{t=0}$. Since $\delta_0(h') = 0$ and $q|_{t=0} \in x_2\mathbf{C}[x_1, x_2]$, it follows that $q|_{t=0} = x_1^{2l}x_2^{l+1}$, proving the claim.

Note: Recall that the *length* $\lambda(\theta)$ of a tame automorphism θ is by definition the minimal number of triangular automorphisms needed to express θ together with affine automorphisms. Due to Furter [3], λ is a lower semi-continuous function on $\mathrm{GA}_2(\mathbf{C}) = \mathrm{TA}_2(\mathbf{C})$. So, if $\psi \in \mathrm{GA}_2(\mathbf{C}[t])$ and $l \geq 0$

are such that $\lambda(\psi_\alpha) \leq l$ for all $\alpha \in \mathbf{C}^*$, then we have $\lambda(\psi_0) \leq l$. Edo-Poloni [1, §1] remarked that this fails if $n = 3$ because of their example. Edo raised a question whether there exists a counterexample in which ψ_0 is also tame. We claim that the answer is yes if $n = 4$: Let $\psi \in \mathrm{GA}_4(\mathbf{C}[t])$ be the extension of the above ϕ defined by $\psi(x_4) = x_4$. Then, we have $\lambda(\psi_\alpha) \leq 3$ for all $\alpha \in \mathbf{C}^*$, while $\lambda(\psi_0) \leq 4$ by Smith [9]. Moreover, we can show that $\lambda(\psi_0)$ is in fact equal to four (the details will appear in a future paper).

References

- [1] E. Edo and P.-M. Poloni, On the closure of the tame automorphism group of affine three-space, arXiv:math.AG/1403.2843v2.
- [2] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, Vol. 190, Birkhäuser, Basel, Boston, Berlin, 2000.
- [3] J.-P. Furter, On the length of polynomial automorphisms of the affine plane, Math. Ann. **322** (2002), 401–411.
- [4] H. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math. **184** (1942), 161–174.
- [5] W. van der Kulk, On polynomial rings in two variables, Nieuw Arch. Wisk. (3) **1** (1953), 33–41.
- [6] S. Kuroda, Shestakov-Umirbaev reductions and Nagata’s conjecture on a polynomial automorphism, Tohoku Math. J. **62** (2010), 75–115.
- [7] S. Kuroda, Wildness of polynomial automorphisms in three variables, arXiv:math.AC/1110.1466v1.
- [8] I. Shestakov and U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, J. Amer. Math. Soc. **17** (2004), 197–227.
- [9] M. K. Smith, Stably tame automorphisms, J. Pure Appl. Algebra **58** (1989), 209–212.

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